MEAN FIELD CONDITIONS FOR COALESCING RANDOM WALKS

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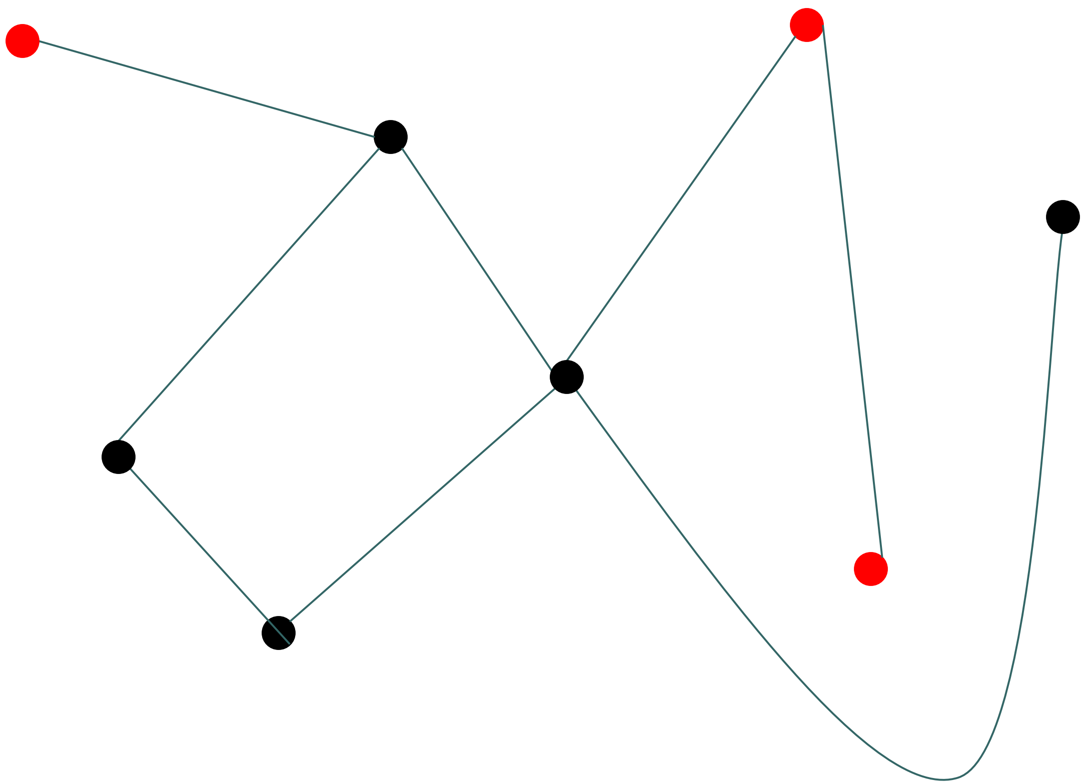
WHAT IS THE VOTER MODEL?



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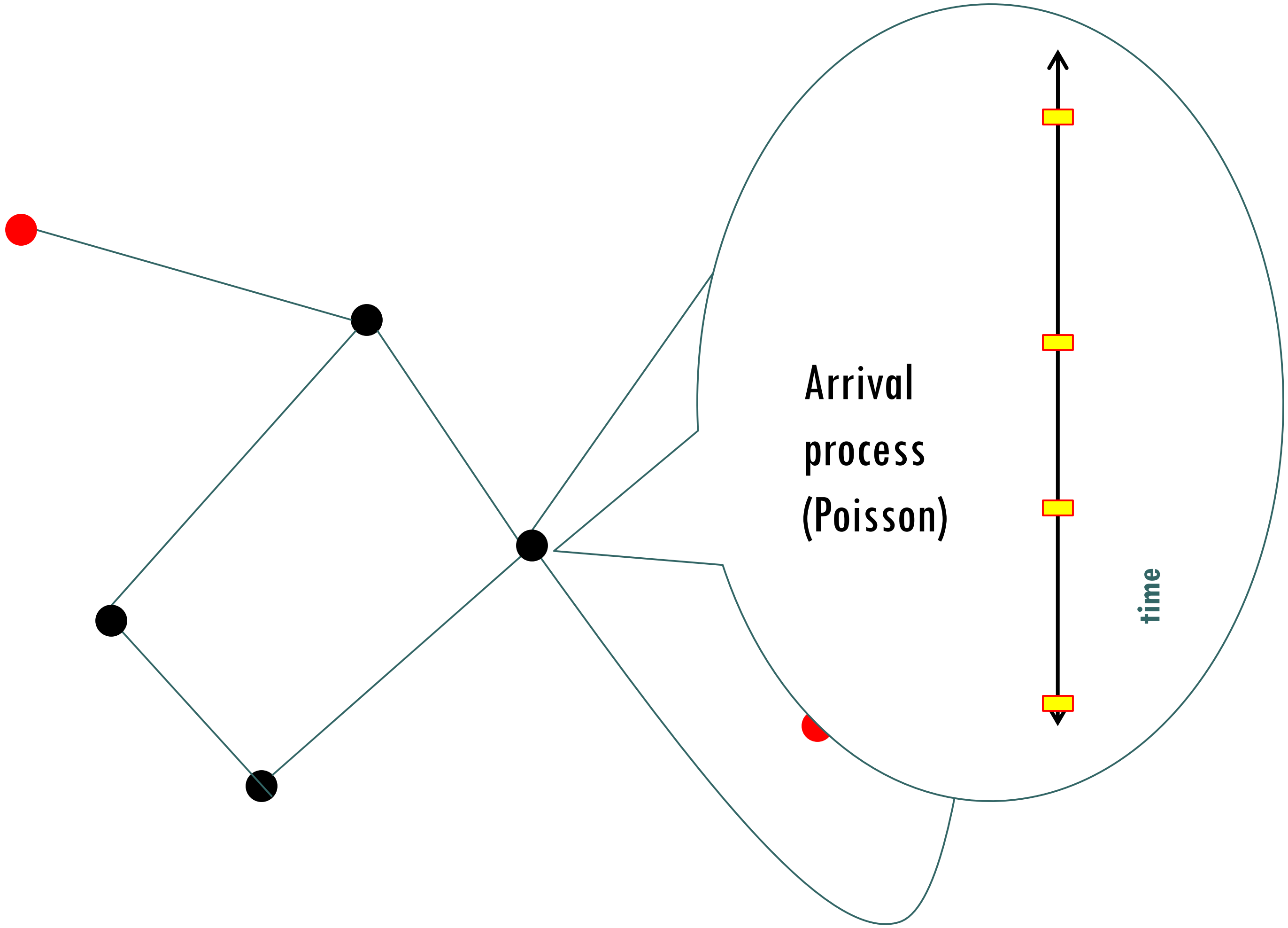

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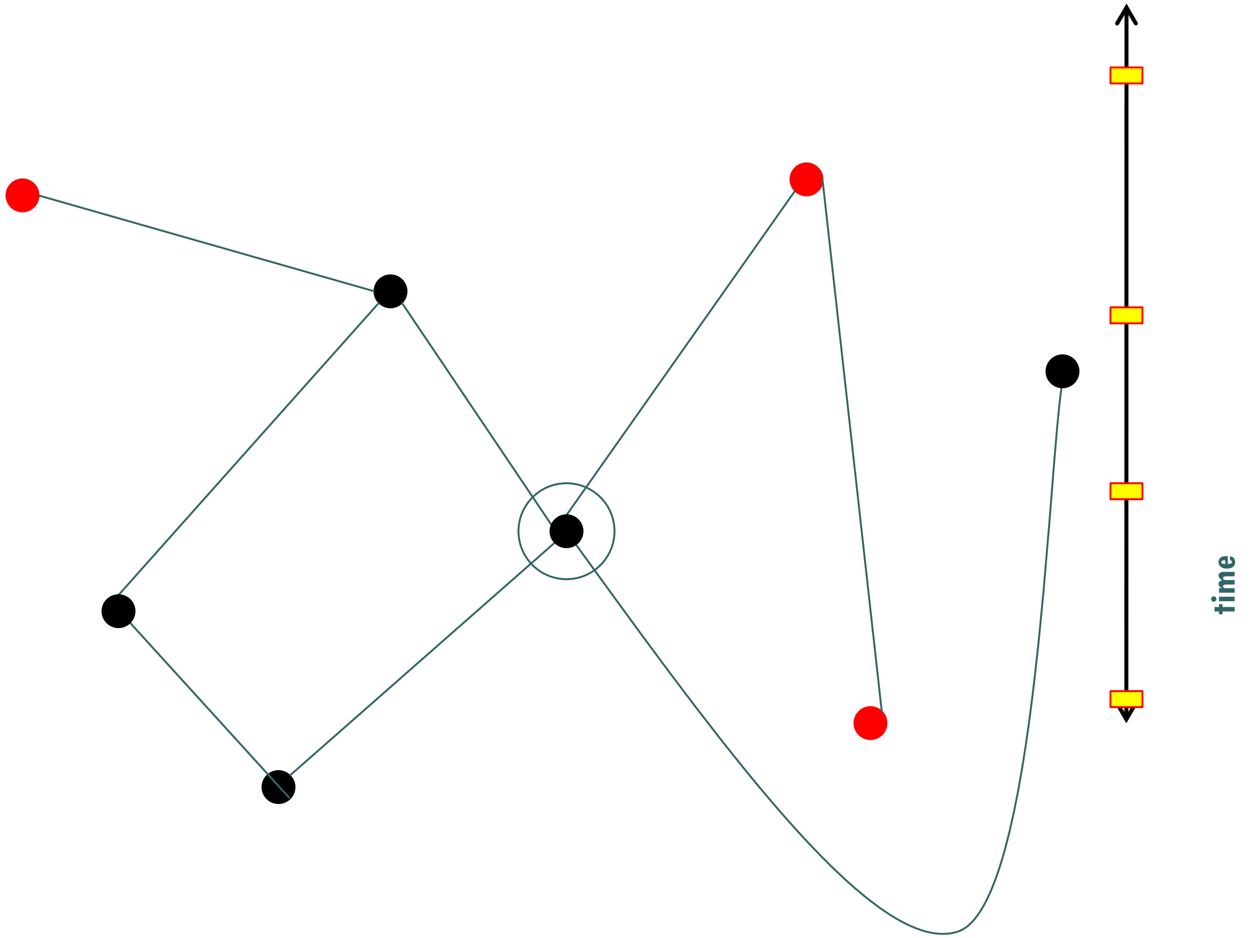

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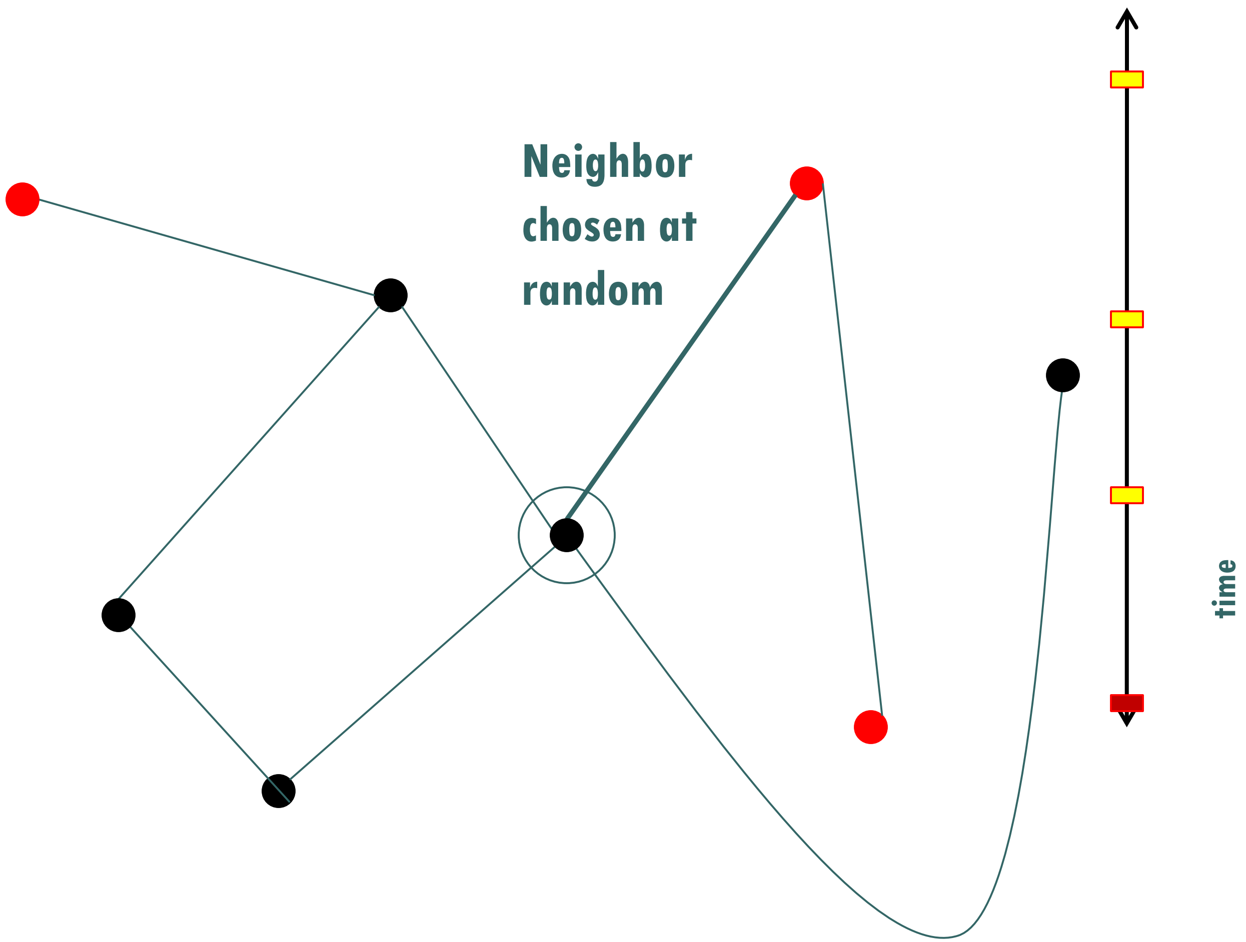

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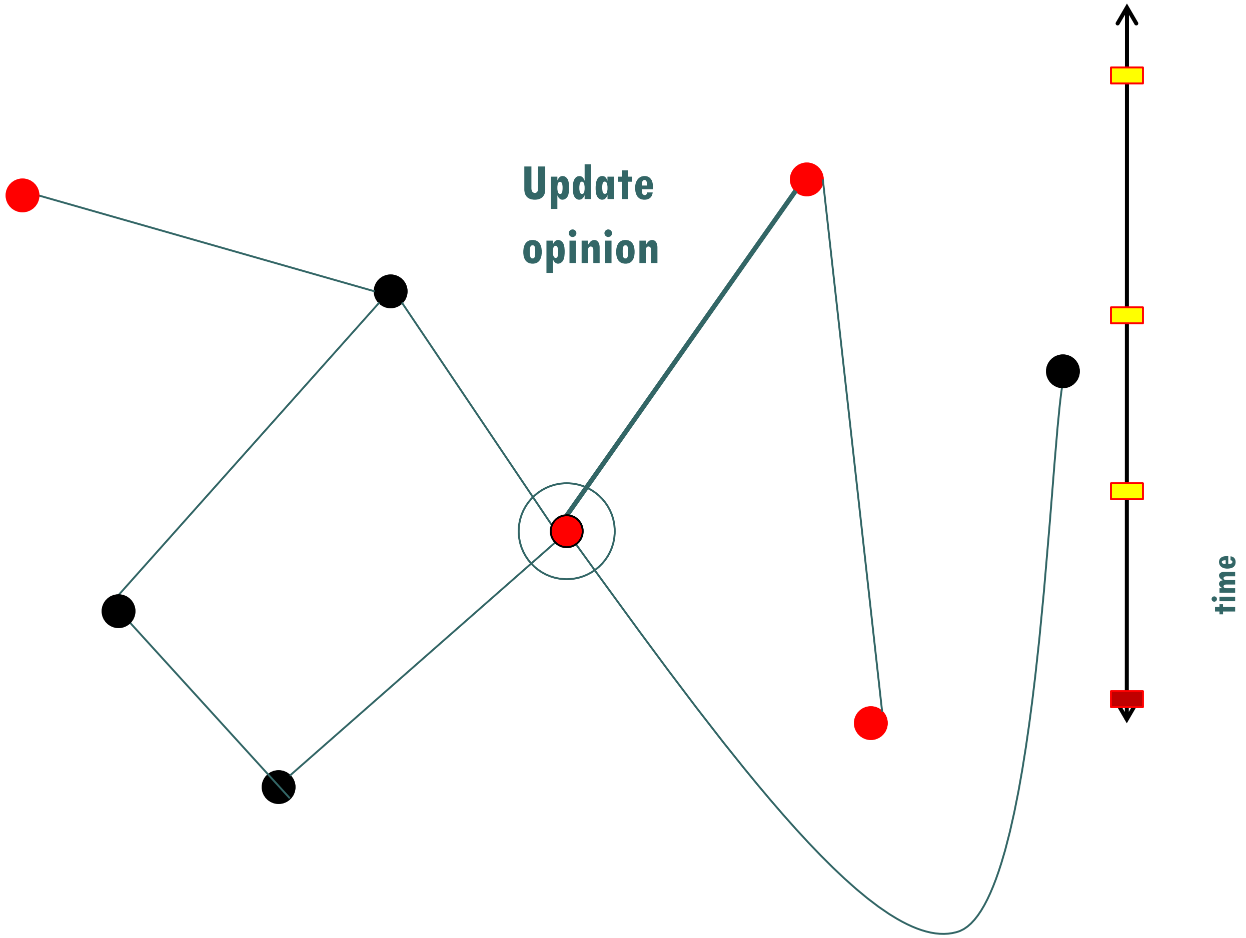
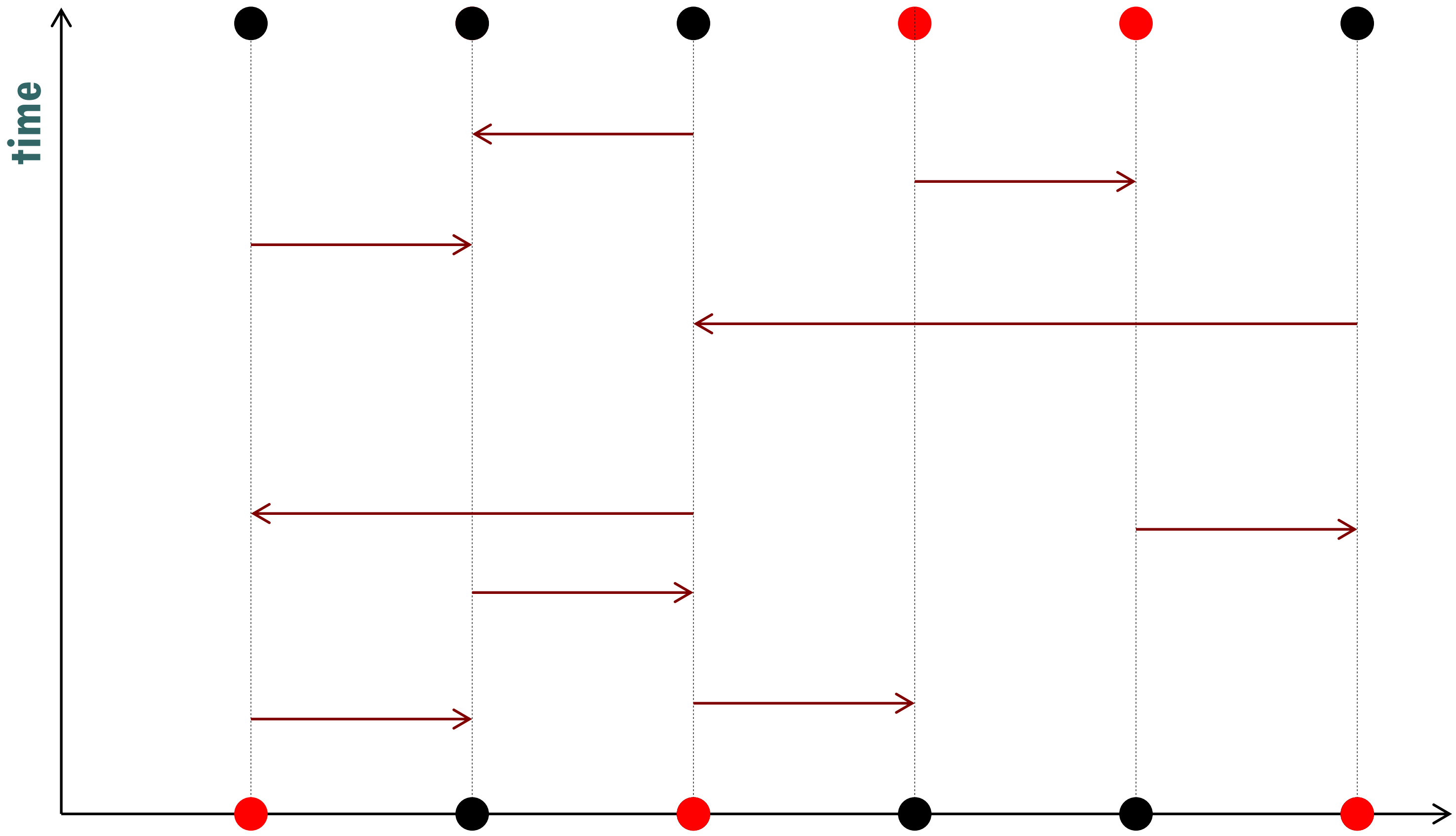


VOTERS, RANDOM WALKERS, AND DUALITY

The original motivation for studying coalescing random walks is the voter model.

DUALITY



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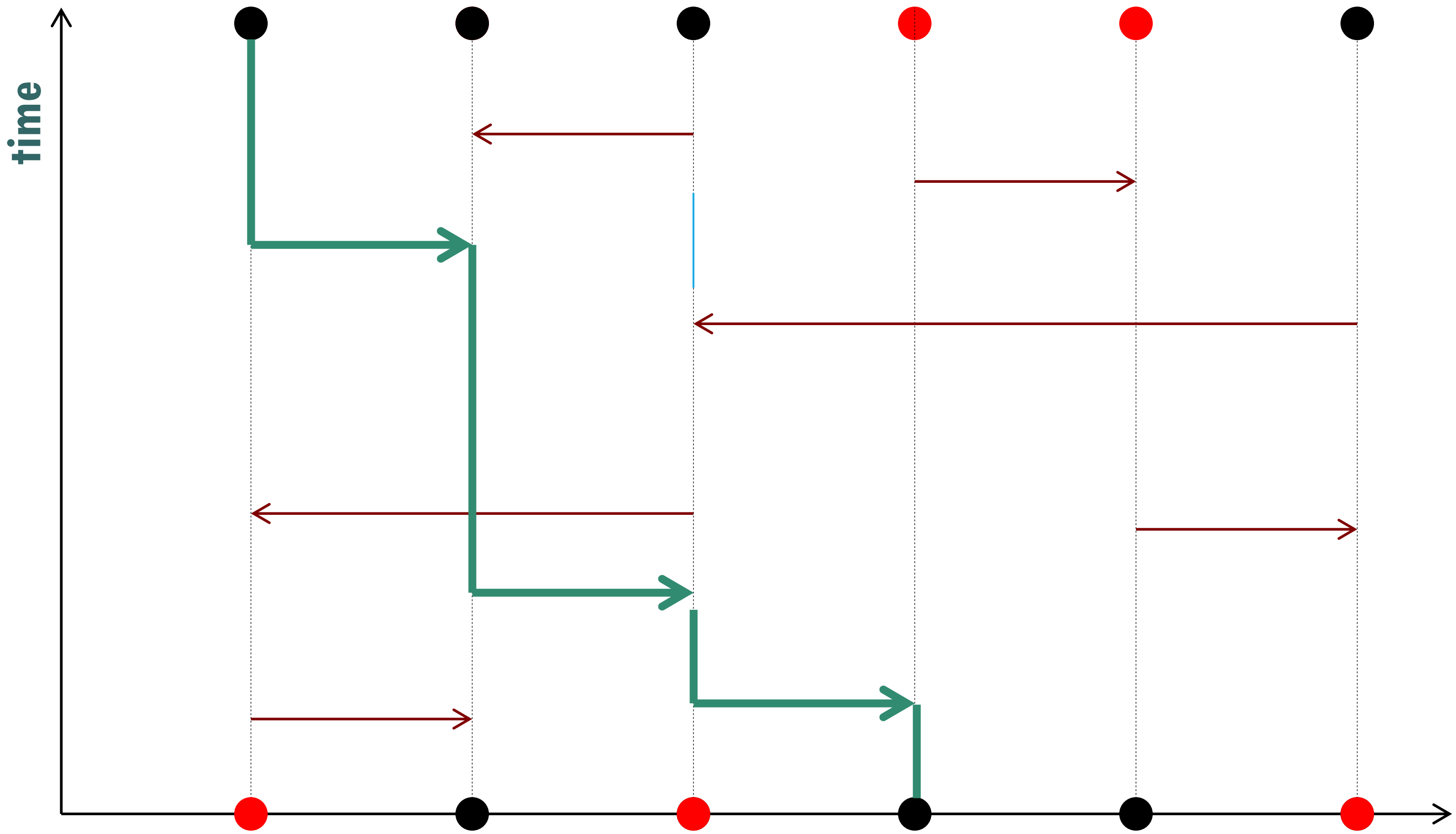

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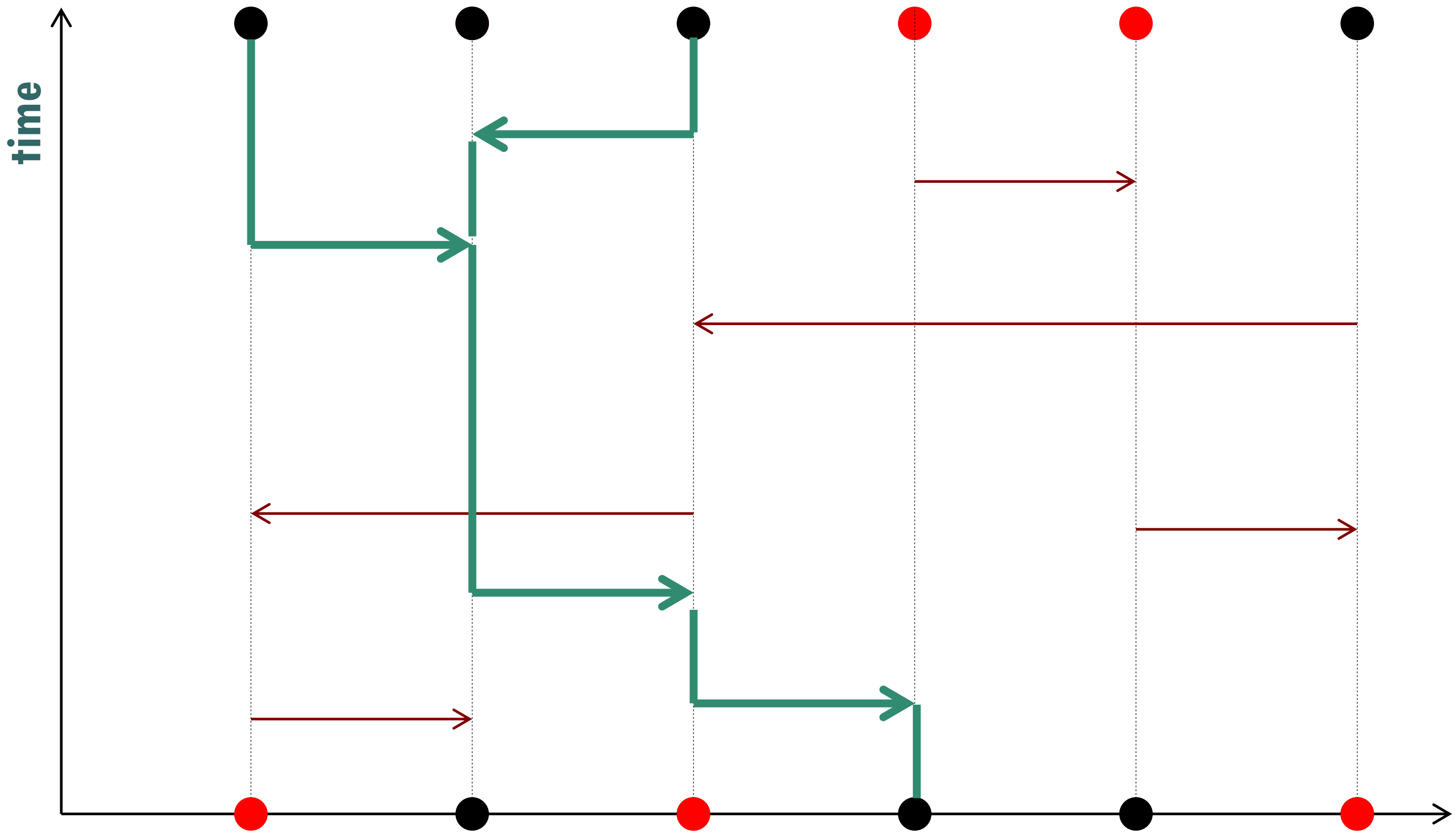

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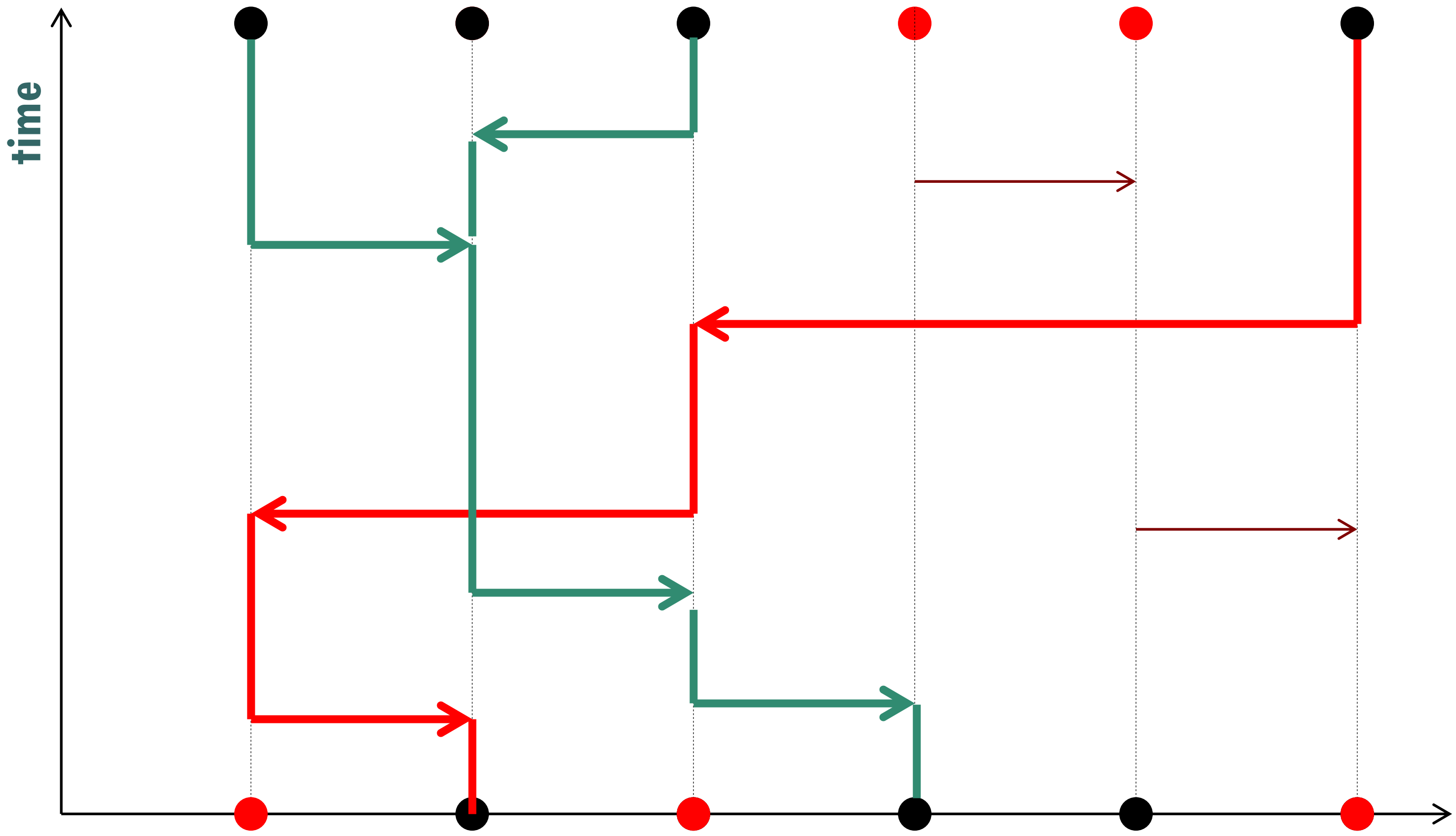

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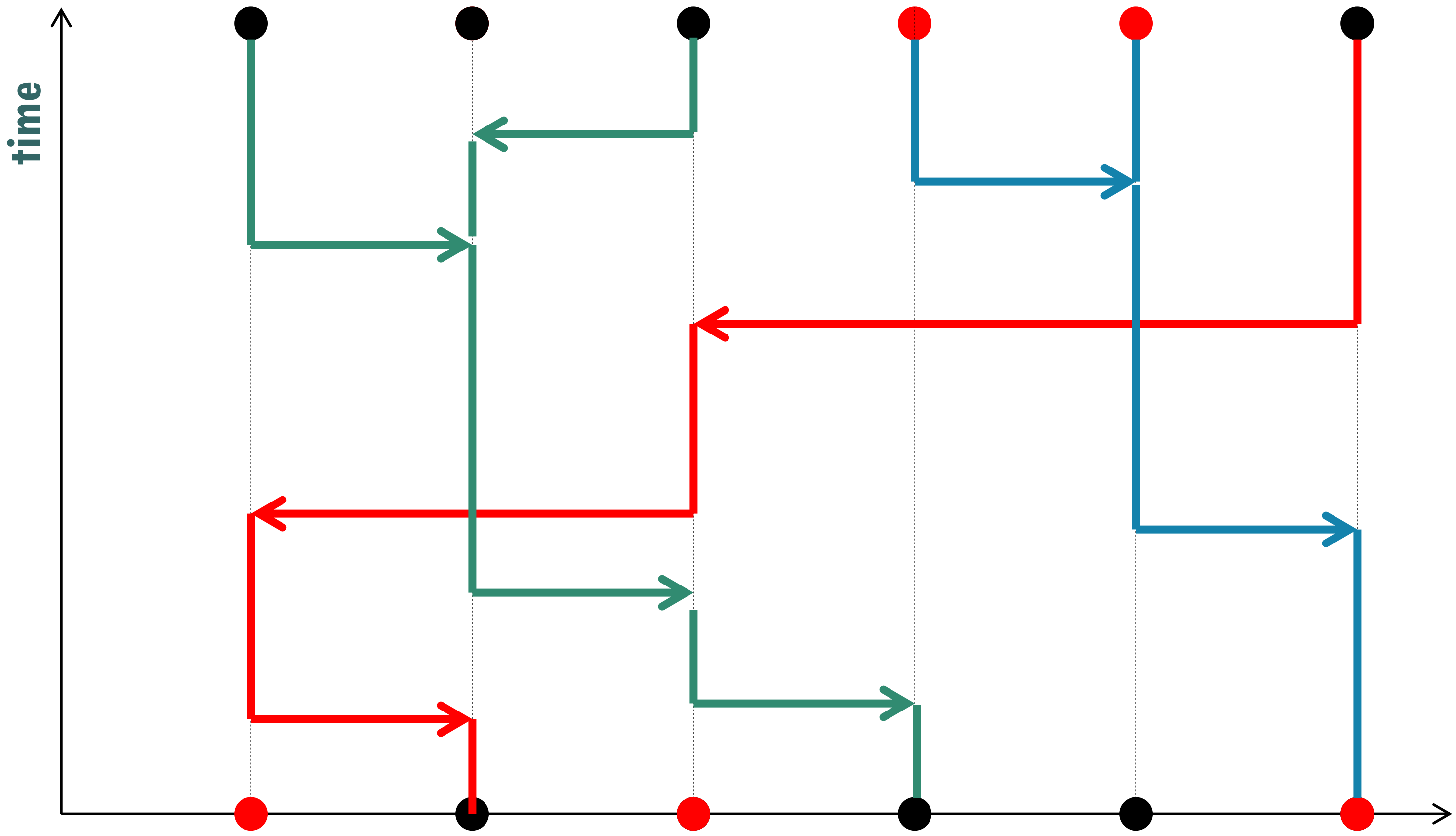

DUALITY

The upshot is that the voter model is dual to

Coalescing random walks,

the main subject of this talk. Will discuss:

C:= full coalescence time.

Results extend to voters with i.i.d. initial opinions.

MEAN FIELD BEHAVIOR FOR FULL COALESCENCE

The case of the complete graph is easy. Other cases turn out to be similar.

THE COMPLETE GRAPH

System of **n** coalescing random walks on a complete graph $\mathbf{K_n}$.

(ie. all transition rates equal to 1).

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Time to move from k to $k-1$ particles:

$$\mathbb{P}(Z_k \geq t) = e^{-\binom{k}{2}t}$$

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i.e. exponential with mean $\binom{k}{2}^{-1}$.

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$$C = \sum_{k=2}^{n} Z_k$$

$\{Z_k\}_k$ independent, $Z_k =_d \exp\left(\frac{1}{\binom{k}{2}}\right)$.

THE COMPLETE GRAPH

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(ie. all transition rates equal to 2).

$$C \overset{\sim}{=} \sum_{k=2}^{\cancel{n}\ \infty} Z_k$$

$$\mathbb{E}(C) \sim 2 = 2\ \mathbb{E}(\text{Meet of } 2)$$

A RESULT BY COX

<u>Cox'91:</u> Consider CRW based on simple random walk in

$$(\mathbb{Z}_n)^d$$

where **d is at least 2**. Then:

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indep. exponentials, means $= \binom{k}{2}^{-1}$

$\mathbb{E}$(Meeting time of 2)

A PROBLEM FROM ALDOUS AND FILL

Prove that this is universal over large transitive graphs

with **relaxation time** small.

A PROBLEM FROM ALDOUS AND FILL

PROBLEM:

$$\frac{C}{\mu_{G_n}} \xrightarrow{n \to +\infty} \sum_{k \geq 2} Z_k$$

Z_k indep. exponentials mean $\binom{k}{2}^{-1}$.

$$\underbrace{\mathbb{E}(\text{Meeting time of 2}) \gg t_{rel}^{G_n}}_{\text{ASSUMPTION}}$$

A PROBLEM FROM ALDOUS AND FILL

Prove that this is universal over large transitive graphs

with **relaxation time** smaller than expected meeting time.

Some assumption is needed: no mean field behavior for star graphs or one-dimensional cycles.

A MORE GENERAL PROBLEM BY DURRETT

In *Random Graph Dynamics* Durrett studies the same kind of problem over certain **random graphs.**

Those have **power law degrees** and are "very non transitive" in many ways.

Nevertheless, D. obtains some partial results in the direction of **<u>universality of mean field behavior</u>**.

MAIN RESULTS

Mean field behavior is indeed very general. We give two results.

A THEOREM FOR TRANSITIVE GRAPHS

$\{Q_n\}_n$: sequence of reversible, transitive chains on finite spaces.

$\{M_n\}_n$: expected meeting time of 2 indep. Q_n-walks

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[$t_{mix,n}$ = mixing time]

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A THEOREM FOR TRANSITIVE CHAINS

$$\frac{C_n}{M_n} \xrightarrow[n \to +\infty]{} \sum_{k \geq 2} Z_k$$

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[Aldous/Fill: $t_{rel,n}/M_n \to 0$.]

A THEOREM FOR GENERAL CHAINS

We also have a theorem not requiring transitivity or reversibility, with messier assumptions.

It covers the random graphs of Durrett + many other examples (eg. supercritical percolation in 3 or more dimensions).

<u>There certainly is room for improvement here.</u>

MAIN PROOF IDEAS

Exponential hitting times, with good error bounds + quantiles + control of big bang phase.

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COMPARE WITH COMPLETE GRAPH

The random variables Z_k have a clearly defined meaning in the complete graph case.

Time to move from k to $k-1$ particles:

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BASIC IDEA: prove similar result for general chains.

CONNECTION WITH HITTING TIMES

k random walks on V $\Rightarrow$ one random walk on V^k (product)

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$\Delta_k = \{(x_i)_{i=1}^k : \exists\, 1 \leq i < j \leq k,\ x_i = x_j\}$

EXPONENTIAL HITTING TIMES

Well-known "metatheorem" (Aldous, Aldous/Brown,...).

$P =$ chain on Ω, $A \subset \Omega$ with π stationary, $\mathbb{E}_\pi(\tau_A) \gg t^P_{mix}$.

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$$\Delta_k = \{(x_i)_{i=1}^{k} : \exists\, 1 \le i < j \le k,\ x_i = x_j\}$$

(larger k)

$$\mathbb{P}_\pi\left(T_k / \mathbb{E}_\pi(T_k) > t\right) \approx e^{-t},$$

WHAT IS MISSING?

EXPONENTIALS

$$\frac{C}{n} = \sum_{k \geq 2} \frac{T_k}{n} \stackrel{?}{\simeq} \sum_{k \geq 2} z_k$$

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EXPONENTIALS

EXPECTATION OF T_2

HITTING OF

$\Delta_k = \{(x_i)_{i \leq k} : \exists i < j \; x_i = x_j\}$

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EXPONEN-TIALS

EXPECTATION OF T_2

HITTING OF $\Delta_k = \{(x_i)_{i\leq k} : \exists i<j \; x_i = x_j\}$

Problem # 1 Better estimates for tail of T_k.

WHAT IS MISSING?

EXPONENTIALS

$$\frac{C}{n} = \sum_{k \geq 2} \frac{T_k}{n} \simeq ? \sum_{k \geq 2} Z_k$$

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Better estimates for tail of T_k.

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EXPECTATION OF T_2

HITTING OF $\Delta_k = \{(x_i)_{i \leq k} : \exists i < j \; x_i = x_j\}$

Problem # 2

Need to consider non-stationary starts.

WHAT IS MISSING?

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EXPECTATION OF T_2

HITTING OF $\Delta_k = \{(x_i)_{i \leq k} : \exists\, i < j \;\; x_i = x_j\}$

Problem # 3

Must show

$$\mathbb{E}(T_k) \approx n \Big/ \binom{k}{2}$$

WHAT IS MISSING?

EXPONENTIALS

$$\frac{c}{n} = \sum_{k \geq 2} \frac{T_k}{n} \stackrel{?}{\simeq} \sum_{k \geq 2} Z_k$$

EXPECTATION OF T_2

HITTING OF

$\Delta_k = \{(x_i)_{i \leq k} : \exists i < j \; x_i = x_j\}$

Problem # 4

k "big" => not exp.

(Big Bang)

EXPONENTIAL HITTING TIMES (NEW)

Sharper theorem.

$P =$ chain on Ω, $A \subset \Omega$ with

π stationary ; $\varepsilon = O\left(\left(\frac{t_{mix}}{\mathbb{E}_\pi(\tau_A)}\right)^{1/3}\right) << 1$

1) $\mathbb{P}_x\left(\frac{\tau_A}{\mathbb{E}_\pi(\tau_A)} > t\right) \leq (1+\varepsilon)\, e^{-\frac{t}{1+\varepsilon}}$

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2) $\mathbb{P}_x\left(\frac{\tau_A}{\mathbb{E}_\pi(\tau_A)} > t\right) \geq (1-\varepsilon)\, e^{-\frac{t}{1-\varepsilon}}$

EXPONENTIAL HITTING TIMES (NEW)

$\hat{\varepsilon} = \varepsilon + \mathbb{P}_x\left(\tau_A < \varepsilon\, \mathbb{E}_\pi(\tau_A)\right)$

Sharper theorem.

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EXPONENTIAL HITTING TIMES (NEW)

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$P =$ chain on Ω, $A \subset \Omega$ with

π stationary; $\varepsilon = \Omega\left(\left(\frac{t_{mix}}{\mathbb{E}_\pi(\tau_A)}\right)^{1/3}\right) << 1$

3) ε-quantile of $\tau_A \approx \varepsilon \mathbb{E}_\pi(\tau_A)$

APPLICATION TO $E(T_K)$

Recall T_k = Hitting time of

$$\Delta_k = \left\{ (x_i)_{i=1}^{k} : \exists\, \underset{x_i = x_j}{i < j} \right\}$$

APPLICATION TO E(T_K)

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$$\Delta_k = \left\{ (x_i)_{i=1}^{k} : \exists\, \underset{x_i = x_j}{i<j} \right\}$$

$$\Rightarrow \mathbb{P}_\pi\left(T_k \leq \varepsilon\, \mathbb{E}(T_2)\right) \lesssim \binom{k}{2} \varepsilon$$

APPLICATION TO E(T_K)

Also need reverse inequa-lity!

Recall T_k = Hitting time of

$$\Delta_k = \{ (x_i)_{i=1}^{k} : \exists\, i<j \; x_i = x_j \}$$

$$\Rightarrow \mathbb{P}_\pi(T_k \leq \varepsilon\, \mathbb{E}(T_2)) \lesssim \binom{k}{2} \varepsilon$$

APPLICATION TO E(T_K)

$$\mathbb{P}_\pi\left(T_k \leq \varepsilon\, \mathbb{E}(T_2)\right) \gtrsim \binom{k}{2} \varepsilon$$

$$- O(k^4)\, \mathbb{P}_\pi\left(\begin{array}{l} T_2^{1,2} \leq \varepsilon\, \mathbb{E}(T_2), \\ T_2^{2,3} \leq \varepsilon\, \mathbb{E}(T_2) \end{array} \right)$$

BOUNDING CORRELATIONS (TRANSITIVE)

$$\mathbb{P}_{\pi^{\otimes 3}}\left(T^{1,2} \leq t,\ T^{2,3} \leq t\right)$$

$$\leq 2\, \mathbb{P}_{\pi^{\otimes 2}}\left(T^{1,2} \leq t\right)^2$$

(on blackboard!)

THE END

Thanks for your attention.

Here is a link to the paper.