First-order logic:
First-order logical equivalence.
Negation of first-order formulae.

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Logical equivalence in first order logic

Formulae A and B are logically equivalent, denoted $A \equiv B$, iff

$$A \models B \text{ and } B \models A.$$

Equivalently, $A \equiv B$ iff

$$\models A \leftrightarrow B.$$

For example, any first-order instance of a pair of equivalent propositional formulae is a pair of logically equivalent formulae.

Some basic properties of logical equivalence:

1. $A \equiv A$
2. If $A \equiv B$ then $B \equiv A$.
3. If $A \equiv B$ and $B \equiv C$ then $A \equiv C$.
4. If $A \equiv B$ then $\neg A \equiv \neg B$, $\forall x A \equiv \forall x B$, and $\exists x A \equiv \exists x B$.
5. If $A_1 \equiv B_1$ and $A_2 \equiv B_2$ then $A_1 \circ A_2 \equiv B_1 \circ B_2$ where $\circ$ is any of $\wedge, \vee, \rightarrow$ and $\leftrightarrow$.

Some logical equivalences involving quantifiers

- $\neg\forall x A \equiv \exists x \neg A$.
- $\neg\exists x A \equiv \forall x \neg A$.
- $\forall x A \equiv \neg\exists x \neg A$.
- $\exists x A \equiv \neg\forall x \neg A$.
- $\exists x \exists y A \equiv \exists y \exists x A$.
- $\forall x \forall y A \equiv \forall y \forall x A$.

NB: $\forall x \exists y A \not\equiv \exists y \forall x A$. Why?

For instance, "For every integer x there is an integer y such that $x < y$" is true, but it does not imply "There is an integer y such that for every integer x, $x < y$.", which is false.

More logical equivalences and non-equivalences

- $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$
- $\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$
- $\forall x(P(x) \to Q(x)) \not\equiv \forall x P(x) \to \forall x Q(x)$
- $\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$
- $\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$
- $\exists x(P(x) \to Q(x)) \stackrel{?}{\equiv} \exists x P(x) \to \exists x Q(x)$

Negating first-order formulae

Using appropriate equivalences, all negations in a first-order formula can be driven inwards, until they reach atomic formulae.

For example, to negate

"For every car, there is a driver who, if (s)he can start it, then (s)he can stop it."

we first translate it to first-order logic:

$$\forall x(\mathrm{Car}(x) \rightarrow \exists y(\mathrm{Driver}(y) \wedge (\mathrm{Start}(x, y) \rightarrow \mathrm{Stop}(x, y)))).$$

Negating first-order formulae: example cont'd

Now negating:

$$\neg\forall x(\mathrm{Car}(x) \to \exists y(\mathrm{Driver}(y) \wedge (\mathrm{Start}(x,y) \to \mathrm{Stop}(x,y))))$$

$$\equiv \exists x\neg(\mathrm{Car}(x) \to \exists y(\mathrm{Driver}(y) \wedge (\mathrm{Start}(x,y) \to \mathrm{Stop}(x,y))))$$

$$\equiv \exists x(\mathrm{Car}(x) \wedge \neg\exists y(\mathrm{Driver}(y) \wedge (\mathrm{Start}(x,y) \to \mathrm{Stop}(x,y))))$$

$$\equiv \exists x(\mathrm{Car}(x) \wedge \forall y\neg(\mathrm{Driver}(y) \wedge (\mathrm{Start}(x,y) \to \mathrm{Stop}(x,y))))$$

$$\equiv \exists x(\mathrm{Car}(x) \wedge \forall y(\neg\mathrm{Driver}(y) \vee \neg(\mathrm{Start}(x,y) \to \mathrm{Stop}(x,y))))$$

$$\equiv \exists x(\mathrm{Car}(x) \wedge \forall y(\neg\mathrm{Driver}(y) \vee (\mathrm{Start}(x,y) \wedge \neg\mathrm{Stop}(x,y)))).$$

Since $\neg A \vee B \equiv A \to B$, the last formula is equivalent to

$$\exists x(\mathrm{Car}(x) \wedge \forall y(\mathrm{Driver}(y) \to (\mathrm{Start}(x,y) \wedge \neg\mathrm{Stop}(x,y)))).$$

Negating first-order formulae: example completed

Thus, the negation of the sentence

"For every car, there is a driver who, if (s)he can start it, then (s)he can stop it."

formalized in first-order logic as

$$\forall x(\mathrm{Car}(x) \rightarrow \exists y(\mathrm{Driver}(y) \wedge (\mathrm{Start}(x, y) \rightarrow \mathrm{Stop}(x, y))))$$

is equivalent to

$$\exists x(\mathrm{Car}(x) \wedge \forall y(\mathrm{Driver}(y) \rightarrow (\mathrm{Start}(x, y) \wedge \neg\mathrm{Stop}(x, y))))$$

which, translated back to natural language, reads:

There is a car such that every driver can start it and cannot stop it.

Negating restricted quantifiers

$$\neg\forall x(P(x) \to A) \quad \equiv \quad \exists x(P(x) \wedge \neg A)$$
$$\neg\exists x(P(x) \wedge A) \quad \equiv \quad \forall x(P(x) \to \neg A),$$

and hence:

$$\neg\forall x \in \mathbf{X}(A) \quad \equiv \quad \exists x \in \mathbf{X}(\neg A)$$
$$\neg\exists x \in \mathbf{X}(A) \quad \equiv \quad \forall x \in \mathbf{X}(\neg A)$$