נתונה המשוואה $y^2u_{xx}+2xyu_{xy}+\alpha x^2u_{yy}-u_x=0$, כאשר $x,y>0$.

א) (5 נקודות) עבור אלו ערכים של α המשוואה פרבולית, היפרבולית, אליפטית? מהי הצורה הקנונית הצפויה?

ב) (10 נקודות) עבור $\alpha=0$ הביאו את המשוואה לצורה קנונית.

<u>פתרון:</u> א) המקדמים: $A=y^2,\quad B=xy,\quad C=\alpha x^2$,

הדיסקרימיננטה: $\Delta=x^2y^2-\alpha x^2y^2=x^2y^2(1-\alpha)$

המשוואה היפרבולית כאשר $\Delta>0 \;\Leftarrow\; (1-\alpha)>0 \;\Leftarrow\; \alpha<1$,

הצורה הקנונית: $v_{st}+F(s,t,v,v_s,v_t)=0$.

המשוואה פרבולית כאשר $\Delta=0 \;\Leftarrow\; (1-\alpha)=0 \;\Leftarrow\; \alpha=1$,

הצורה הקנונית: $v_{tt}+F(s,t,v,v_s,v_t)=0$.

המשוואה אליפטית כאשר $\Delta<0 \;\Leftarrow\; (1-\alpha)<0 \;\Leftarrow\; \alpha>1$,

הצורה הקנונית: $v_{ss}+v_{tt}+F(s,t,v,v_s,v_t)=0$.

ב) עבור $\alpha=0$ המשוואה היא: $y^2u_{xx}+2xyu_{xy}-u_x=0$.

נפתור מד"ר

$$y' = \frac{xy \pm \sqrt{(xy)^2}}{y^2} = \frac{xy \pm xy}{y^2}$$

$$y' = \frac{2x}{y} \Rightarrow \int y dy = 2\int x dx \Rightarrow \frac{y^2}{2} = \frac{2x^2}{2} + c \Rightarrow 2x^2 - y^2 = const$$

$$y' = 0 \Rightarrow y = const$$

נגדיר $s(x,y) = -2x^2 + y^2, \quad t(x,y) = y$ ונסמן

$.u(x,y) = u\left(x(s,t), y(s,t)\right) = v\left(s(x,y), t(x,y)\right)$

הנגזרות:

$$u_x = -4xv_s,$$
$$u_{xx} = v_{ss}(-4x)^2 - 4v_s = 16x^2 v_{ss} - 4v_s$$
$$u_{xy} = -8xyv_{ss} - 4xv_{st},$$
$$u_y = 2yv_s + v_t,$$

ע"י הצבה במשוואה: $y^2\left(16x^2 v_{ss} - 4v_s\right) + 2xy\left(-8xyv_{ss} - 4xv_{st}\right) - \left(-4xv_s\right) = 0$

וכיוון ש $y = t, \quad x^2 = \frac{t^2 - s}{2}$

הצורה הקנונית: $v_{st} = \frac{(x - y^2)}{2x^2 y} v_s = 0 \quad \Leftarrow \quad v_{st} = \frac{\sqrt{t^2 - s} - \sqrt{2}t^2}{\sqrt{2}t(t^2 - s)} v_s = 0$

שאלה 2 :

נתונה משוואה $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=1$.

א) (5 נקודות) רשמו את סוג המשוואה.

ב) (20 נקודות) בהנתן החלפת המשתנים: $t(x,y)=x, \quad s(x,y)=\frac{y}{x}$,

מצאו צורה קנונית של המשוואה.

ג) (10 נקודות) מצאו פתרון כללי למד"ח.

פתרון

א) $A=x^2, \quad B=xy, \quad C=y^2 \quad \Rightarrow \quad \Delta=0$ המשוואה פרבולית.

ב) נתון: $s(x,y)=\frac{y}{x}, \quad t(x,y)=x, \quad u(x,y)=v(s,t)$

נגזור: $u_x=v_s\left(\frac{-y}{x^2}\right)+v_t, \quad u_{xx}=v_{ss}\left(\frac{-y}{x^2}\right)^2+2v_{st}\left(\frac{-y}{x^2}\right)+v_s\left(\frac{2y}{x^3}\right)+v_{tt}$

$$u_{xy}=v_{ss}\left(\frac{-y}{x^2}\right)\left(\frac{1}{x}\right)+v_s\left(\frac{-1}{x^2}\right)+v_{ts}\left(\frac{1}{x}\right), \quad u_y=v_s\left(\frac{1}{x}\right), \quad u_{yy}=v_{ss}\left(\frac{1}{x}\right)^2$$

נציב במד"ח:

$$\left(\frac{y^2}{x^2}v_{ss}-2yv_{st}+\frac{2yv_s}{x}+x^2v_{tt}\right)+\left(v_{ss}\left(\frac{-2y^2}{x^2}\right)+v_s\left(\frac{-2y}{x}\right)+2yv_{ts}\right)+\left(\frac{y^2}{x^2}v_{ss}\right)=0$$

צורה קנונית $v_{tt}=\frac{1}{t^2}$.

ג) פתרון:

$$v_{tt}=\frac{1}{t^2} \quad \Rightarrow \quad \int v_{tt}dt=\int\frac{1}{t^2}dt \quad \Rightarrow \quad v_t=-\frac{1}{t}+f(s) \quad \Rightarrow \quad \int v_t dt=\int\left(-\frac{1}{t}+f(s)\right)dt$$

$$\Rightarrow \quad v=-\ln|t|+tf(s)+g(s) \quad \Rightarrow \quad u=-\ln|x|+xf\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$$

בהצלחה!

$$\frac{\partial^2 z}{\partial x^2}\sin^2 x - 2y\sin x\frac{\partial^2 z}{\partial x\partial y} + y^2\cdot\frac{\partial^2 z}{\partial y^2} = 0.$$

$a=\sin^2 x$, $2b=-2y\sin x$, $c=y^2$

$a=\sin^2 x$, $b=-y\sin x$, $c=y^2$

$$\Delta = b^2-ac = (-y\sin x)^2 - \sin^2 x\cdot y^2 = y^2\cancel{\sin^2 x} - y^2\cancel{\sin^2 x} = 0$$

$\Delta=0$ ולכן המד״ח מסוג פרבולית.

$a\,dy^2 - 2b\,dx\,dy + c\,dx^2 = 0.$ משוואה אופיינית!

$\sin^2 x\,dy^2 + 2y\sin x\,dx\,dy + y^2\,dx^2 = 0.\ /:dx^2 \neq 0$

$$\sin^2 x\cdot\left(\frac{dy}{dx}\right)^2 + 2y\cdot\sin x\cdot\frac{dy}{dx} + y^2 = 0.$$

$$\frac{dy}{dx} = \frac{-y\sin x \pm\sqrt{0}}{\sin^2 x} = \frac{-y\cancel{\sin x}}{\sin^{\cancel{2}} x} = \frac{-y}{\sin x}.$$

$$\frac{dy}{dx} = \frac{-y}{\sin x}\quad / y\neq 0,\ \sin x\neq 0$$

$$\frac{dy}{y} = -\frac{1}{\sin x}dx\ /\int.$$

$$\ln y = \int -\frac{1}{\sin x}dx.$$

$$\ln y = \int\frac{-\sin x}{\sin^2 x}dx \;\Rightarrow\; \ln y = \int\frac{-\sin x}{1-\cos^2 x}dx.$$

זהות טריגונומטרית
$\sin^2 x = 1-\cos^2 x$

נציב
$\cos x = t$
$-\sin x\,dx = dt.$

$$\ln y = \int\frac{1}{1-t^2}dt.$$

$$\int\frac{1}{1-t^2}dt\ ,\quad \frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t}$$

$$1 = A(1+t) + B(1-t)$$

$2A = 1\ /:2$ נציב $t=1$

$\boxed{A=\frac{1}{2}}$

$1 = 2B \Rightarrow \boxed{B=\frac{1}{2}}$ נציב $t=-1$

$$\frac{1}{1-t^2} = \frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t} \Rightarrow \frac{1}{1-t^2} = \frac{1}{2}\cdot\frac{1}{1-t} + \frac{1}{2}\cdot\frac{1}{1+t}$$

$$\int \frac{1}{1-t^2}\,dt = \frac{1}{2}\int \frac{1}{1-t}\,dt + \frac{1}{2}\int \frac{1}{1+t}\,dt$$

$$\int \frac{1}{1-t^2}\,dt = -\frac{1}{2}\ln(1-t) + \frac{1}{2}\ln(1+t) = \frac{1}{2}\left[\ln(1+t) - \ln(1+t)\right]$$

$$= \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) + C$$

$$\Downarrow$$

$$\boxed{\int \frac{-1}{\sin x}\,dx = \frac{1}{2}\ln\left(\frac{1+\cos x}{1-\cos x}\right) + C}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{1+\cos x}{1-\cos x}\right) + C_1 \quad / \cdot 2$$

$$2\ln y = \ln\left(\frac{1+\cos x}{1-\cos x}\right) + C_1$$

$$\ln y^2 = \ln\left(\frac{1+\cos x}{1-\cos x}\right) + C_1 \quad / e \qquad , \quad e^{C_1} = C_1 \text{ (סמן)}$$

$$y^2 = \left(\frac{1+\cos x}{1-\cos x}\right)\cdot C_1$$

נעזר בזהות טריגונומטרית לזווית כפולה

$$\frac{1+\cos x}{1-\cos x} = \frac{\cancel{2}\cos^2(\frac{x}{2})}{\cancel{2}\sin^2(\frac{x}{2})} \Leftarrow \qquad \cos x + 1 = 2\cos^2\frac{x}{2} \Leftarrow \cos x = 2\cos^2(\tfrac{x}{2}) - 1$$

$$1 - \cos x = 2\sin^2\frac{x}{2} \Leftarrow \cos x = 1 - 2\sin^2\frac{x}{2} \text{ וגם}$$

$$= \left(\frac{\cos(\frac{x}{2})}{\sin\frac{x}{2}}\right)^2 = \left(ctg(\tfrac{x}{2})\right)^2$$

(זהות טריגונומטרית $ctg\,x = \frac{1}{tg\,x}$)

$$y^2 = \left(ctg(\tfrac{x}{2})\right)^2 \cdot C_1 \Rightarrow y^2 = \left(\frac{1}{tg(\frac{x}{2})}\right)^2 \cdot C_1$$

($\bar{C}_1 = C_1$ (סמן))

$$y^2 = \frac{1}{tg^2\frac{x}{2}}\cdot C_1 \Rightarrow y^2 = \frac{C_1}{tg^2(\frac{x}{2})} \quad /\sqrt{\ }$$

$$y = \frac{C_1}{tg(\frac{x}{2})} \Rightarrow \boxed{C_1 = y \cdot tg(\tfrac{x}{2})}$$

$$\xi(x,y) = y \cdot tg(\tfrac{x}{2})$$

נבחר $\eta(x,y)$ כך ש- $J \neq 0$ נבחר $\eta(x,y) = y$ בהתאמה

$$J(x,y) = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} \frac{y}{2\cos^2(\frac{x}{2})} & tg(\frac{x}{2}) \\ 0 & 1 \end{vmatrix} = \frac{y}{2\cos^2(\frac{x}{2})} \neq 0 \quad \text{כי } y \neq 0$$

עבור $y=0.$

$$\frac{\partial^2 z}{\partial x^2}\cdot \sin^2 x = 0 \quad / : \sin^2 x \neq 0$$

$$\frac{\partial^2 z}{\partial x^2} = 0 \;\Rightarrow\; \frac{\partial}{\partial x}\Big(\overbrace{\frac{\partial z}{\partial x}}^{V \text{ סימון}}\Big) = 0 \;\Rightarrow\; \frac{\partial V}{\partial x} = 0 \;\int dx$$

$$\frac{\partial z}{\partial x} = f_1(y) \int dx \;\Leftarrow\; \frac{\partial z}{\partial x} = V = f_1(y) \;\Leftarrow\; V = f_1(y)$$

$$\boxed{z(x,y) = x\cdot f_1(y) + f_2(y)}$$

נחזור לתרגיל : עבור $y\neq 0$ $J(x,y)\neq 0.$

$$\begin{cases} \xi(x,y) = y\cdot tg\left(\frac{x}{2}\right) \\ \eta(x,y) = y. \end{cases}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{y}{2\cos^2\left(\frac{x}{2}\right)} + \frac{\partial z}{\partial \eta}\cdot 0 = \frac{y}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{y}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi}\cdot tg\left(\frac{x}{2}\right) + \frac{\partial z}{\partial \eta}\cdot 1$$

$$\boxed{\frac{\partial z}{\partial y} = tg\left(\frac{x}{2}\right)\cdot\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{y}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}\right) = \frac{y}{2}\,\frac{\partial}{\partial x}\left(\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}\right) =$$

$$= \frac{y}{2}\left[\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \xi}\right) + \frac{\partial z}{\partial \xi}\cdot\left(\frac{\partial}{\partial x}\left(\frac{1}{\cos^2\left(\frac{x}{2}\right)}\right)\right)\right] =$$

$$\frac{y}{2}\left[\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\left(\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \xi\,\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) + \frac{\partial z}{\partial \xi}\cdot\frac{\sin\left(\frac{x}{2}\right)}{\cos^3\left(\frac{x}{2}\right)}\right] =$$

$$= \frac{y}{2}\left[\frac{1}{\cos^2\left(\frac{x}{2}\right)}\left(\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{y}{2\cos^2\left(\frac{x}{2}\right)} + \frac{\partial^2 z}{\partial \xi\,\partial \eta}\cdot 0\right) + \frac{\sin\left(\frac{x}{2}\right)}{\cos^3\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}\right] =$$

$$= \frac{y}{2}\left[\frac{y}{2\cos^4\left(\frac{x}{2}\right)}\cdot\frac{\partial^2 z}{\partial \xi^2} + \frac{\sin\left(\frac{x}{2}\right)}{\cos^3\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}\right] = \frac{y^2}{4\cos^4\left(\frac{x}{2}\right)}\cdot\frac{\partial^2 z}{\partial \xi^2} + \frac{y}{2}\frac{\sin\left(\frac{x}{2}\right)}{\cos^3\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi},$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{4\cos^4\left(\frac{x}{2}\right)}\cdot\frac{\partial^2 z}{\partial \xi^2} + \frac{y\sin\left(\frac{x}{2}\right)}{2\cos^3\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(tg\left(\frac{x}{2}\right)\cdot\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) =$$

$$= \frac{\partial}{\partial x}\left(tg\left(\frac{x}{2}\right)\cdot\frac{\partial z}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \eta}\right) = tg\left(\frac{x}{2}\right)\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \xi}\right) + \frac{\partial z}{\partial \xi}\cdot\frac{\partial}{\partial x}\left(tg\left(\frac{x}{2}\right)\right)$$

$$+ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \eta}\right) = tg\left(\frac{x}{2}\right)\left[\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot\frac{\partial \eta}{\partial x}\right] + \frac{1}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}$$

$$+ \frac{\partial^2 z}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial^2 z}{\partial \eta \cdot \partial \xi}\cdot\frac{\partial \xi}{\partial x} = tg\left(\frac{x}{2}\right)\left[\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{y}{2\cos^2\left(\frac{x}{2}\right)} + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot 0\right]$$

$$+ \frac{1}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi} + \frac{\partial^2 z}{\partial \eta^2}\cdot 0 + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot\frac{y}{2\cos^2\left(\frac{x}{2}\right)} =$$

$$\frac{y\, tg\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial^2 z}{\partial \xi^2} + \frac{1}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi} + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot\frac{y}{2\cos^2\left(\frac{x}{2}\right)}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{y\, tg\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot\frac{y}{2\cos^2\left(\frac{x}{2}\right)} + \frac{1}{2\cos^2\left(\frac{x}{2}\right)}\cdot\frac{\partial z}{\partial \xi}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(tg\left(\frac{x}{2}\right)\cdot\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) = \frac{\partial}{\partial y}\left(tg\left(\frac{x}{2}\right)\cdot\frac{\partial z}{\partial \xi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \eta}\right)$$

$$= tg\left(\frac{x}{2}\right)\cdot\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \xi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \eta}\right) = tg\left(\frac{x}{2}\right)\left[\frac{\partial^2 z}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot\frac{\partial \eta}{\partial y}\right]$$

$$+ \frac{\partial^2 z}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial^2 z}{\partial \eta \partial \xi}\cdot\frac{\partial \xi}{\partial y} = tg\left(\frac{x}{2}\right)\left[\frac{\partial^2 z}{\partial \xi^2}\cdot tg\left(\frac{x}{2}\right) + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\, 1\right]$$

$$+ \frac{\partial^2 z}{\partial \eta^2}\cdot 1 + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot tg\left(\frac{x}{2}\right) = tg^2\left(\frac{x}{2}\right)\cdot\frac{\partial^2 z}{\partial \xi^2} + tg\left(\frac{x}{2}\right)\cdot\frac{\partial^2 z}{\partial \xi \cdot \partial \eta}$$

$$+ \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}\cdot tg\left(\frac{x}{2}\right) = tg^2\left(\frac{x}{2}\right)\cdot\frac{\partial^2 z}{\partial \xi^2} + 2\, tg\left(\frac{x}{2}\right)\frac{\partial^2 z}{\partial \xi \cdot \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = tg^2\left(\frac{x}{2}\right)\cdot\frac{\partial^2 z}{\partial \xi^2} + 2\, tg\left(\frac{x}{2}\right)\cdot\frac{\partial^2 z}{\partial \xi \cdot \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}}$$

$$\begin{cases} \xi(x,y) = y\cdot tg\left(\frac{x}{2}\right) \\ \eta(x,y) = y \end{cases}$$

$$\Downarrow$$

$$\xi = \eta\cdot tg\left(\frac{x}{2}\right)$$

$$\boxed{tg\left(\frac{x}{2}\right) = \frac{\xi}{\eta}}$$

$$\sin^2 x \cdot \frac{\partial^2 z}{\partial x^2} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} - 2y \sin x \cdot \frac{\partial^2 z}{\partial x \cdot \partial y} = 0$$

$$\sin^2 x \left[\frac{y^2}{4\cos^4(\frac{x}{2})} \cdot \frac{\partial^2 z}{\partial \xi^2} + \frac{y \cdot \sin(\frac{x}{2})}{2\cos^3(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi} \right] + y^2 \cdot \left[tg^2(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi^2} + 2tg(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi \partial \eta}\right.$$

$$\left. + \frac{\partial^2 z}{\partial \eta^2} \right] - 2y \sin x \cdot \left[\frac{y\, tg(\frac{x}{2})}{2\cos^2(\frac{x}{2})} \cdot \frac{\partial^2 z}{\partial \xi^2} + \frac{y}{2\cos^2(\frac{x}{2})} \cdot \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} + \frac{1}{2\cos^2(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi} \right] =$$

$$\frac{\cancel{4}\sin^2(\frac{x}{2}) \cdot \cancel{\cos^2(\frac{x}{2})} \cdot y^2}{\cancel{4}\cos^{\cancel{4}\,2}(\frac{x}{2})} \cdot \frac{\partial^2 z}{\partial \xi^2} + \frac{\cancel{4}^{\,2}\sin^2(\frac{x}{2}) \cdot \cancel{\cos^2(\frac{x}{2})} \cdot y \cdot \sin(\frac{x}{2})}{\cancel{2}\cos^3(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi} + y^2 tg^2(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi^2}$$

$$+ 2y^2 tg(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} + y^2 \cdot \frac{\partial^2 z}{\partial \eta^2} - \frac{\cancel{2} y^2 \sin x\, tg(\frac{x}{2})}{\cancel{2}\cos^2(\frac{x}{2})} \cdot \frac{\partial^2 z}{\partial \xi^2}$$

$$- \frac{\cancel{2} y^2 \sin x}{\cancel{2}\cos^2(\frac{x}{2})} \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} - \frac{\cancel{2} y \sin x}{\cancel{2}\cos^2(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi} = 0$$

$$y^2 \cdot tg^2(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi^2} + \frac{2y \sin^3(\frac{x}{2})}{\cos(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi} + y^2 tg^2(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi^2} + 2y^2 tg(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}$$

$$+ y^2 \cdot \frac{\partial^2 z}{\partial \eta^2} - \frac{2y^2 \sin\frac{x}{2} \cdot \cancel{\cos\frac{x}{2}} \cdot tg(\frac{x}{2})}{\cos^{\cancel{2}}(\frac{x}{2})} \cdot \frac{\partial^2 z}{\partial \xi^2} - \frac{2y^2 \sin\frac{x}{2} \cdot \cancel{\cos\frac{x}{2}}}{\cos^{\cancel{2}}(\frac{x}{2})} \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}$$

$$- \frac{2y \cdot \sin(\frac{x}{2}) \cdot \cancel{\cos(\frac{x}{2})}}{\cos^{\cancel{2}}(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi} = 0$$

$$\cancel{2y^2 tg^2(\tfrac{x}{2}) \cdot \frac{\partial^2 z}{\partial \xi^2} - 2y^2 tg(\tfrac{x}{2}) \frac{\partial^2 z}{\partial \xi^2}} + \frac{2y \sin^3(\frac{x}{2})}{\cos(\frac{x}{2})} \frac{\partial z}{\partial \xi} - \frac{2y \sin(\frac{x}{2})}{\cos(\frac{x}{2})} \cdot \frac{\partial z}{\partial \xi}$$

$$+ \cancel{2y^2 tg(\tfrac{x}{2}) \frac{\partial^2 z}{\partial \xi \cdot \partial \eta} - 2y^2 tg(\tfrac{x}{2}) \frac{\partial^2 z}{\partial \xi \cdot \partial \eta}} + y^2 \cdot \frac{\partial^2 z}{\partial \eta^2} = 0$$

$$y^2 \cdot \frac{\partial^2 z}{\partial \eta^2} + 2y \cdot \sin^2(\tfrac{x}{2}) \cdot tg(\tfrac{x}{2}) \cdot \frac{\partial z}{\partial \xi} - 2y \cdot tg(\tfrac{x}{2}) \cdot \frac{\partial z}{\partial \xi} = 0.$$

$$y^2 \frac{\partial^2 z}{\partial \eta^2} + 2y \cdot tg(\tfrac{x}{2}) \underbrace{(\sin^2(\tfrac{x}{2}) - 1)}_{-\cos^2\frac{x}{2}} \cdot \frac{\partial z}{\partial \xi} = 0$$

$$\eta^2 \cdot \frac{\partial^2 z}{\partial \eta^2} - 2y\, tg(\tfrac{x}{2}) \cos^2(\tfrac{x}{2}) \cdot \frac{\partial z}{\partial \xi} = 0$$

$$\eta^2 \frac{\partial^2 z}{\partial \eta^2} = 2y\, tg(\tfrac{x}{2}) \cdot \cos^2(\tfrac{x}{2}) \cdot \frac{\partial z}{\partial \xi}$$

$$tg(\tfrac{x}{2}) = \frac{\xi}{\eta}$$

(נוסחה טריגונומטרית) $1 + tg^2(\frac{x}{2}) = \frac{1}{\cos^2(\frac{x}{2})}$

$$\frac{\eta^2 + \xi^2}{\eta^2} = \frac{1}{\cos^2(\frac{x}{2})}$$

$$1 + \left(\frac{\xi}{\eta}\right)^2 = \frac{1}{\cos^2(\frac{x}{2})}$$

$$\Downarrow$$

$$\cos^2(\tfrac{x}{2}) = \frac{\eta^2}{\eta^2 + \xi^2}$$

$$\eta^2 \cdot \frac{\partial^2 z}{\partial \eta^2} = 2 \cdot \cancel{\eta} \cdot \frac{\xi}{\cancel{\eta}} \cdot \left(\frac{\eta^2}{\xi^2+\eta^2} \right) \cdot \frac{\partial z}{\partial \xi}$$

$$\eta^2 \cdot \frac{\partial^2 z}{\partial \eta^2} = \frac{2\xi \cdot \eta^2}{(\xi^2+\eta^2)} \cdot \frac{\partial z}{\partial \xi} \quad / : \eta^2 \neq 0.$$

$$\boxed{\frac{\partial^2 z}{\partial \eta^2} = \frac{2\xi}{(\xi^2+\eta^2)} \cdot \frac{\partial z}{\partial \xi}}$$

צורה קנונית של
משו' פרבולית.

<u>מש"ל</u>

שאלה 2: העבר לצורה קנונית $xu_{xx}-yu_{yy}+\frac{1}{2}(u_x-u_y)=0$ $(x,y\neq 0)$.

פתרון: $a=x,b=0,c=-y$.

קודם כל, נבדוק איזה סוג המשוואה: $\Delta=xy$.

נחלק למקרים:

מקרה 1: $\Delta=xy>0$ ולכן היפרבולי.

ע"פ המשוואה האופיינית נקבל:

$$\frac{dy}{dx}=\frac{\pm\sqrt{xy}}{x}=\pm\sqrt{\frac{y}{x}}$$

ע"י פתירת המד"ר נקבל $c_{1,2}=\sqrt{y}\pm\sqrt{x}$ ולכן נקבל $p(x,y)=\sqrt{y}+\sqrt{x},\ q(x,y)=\sqrt{y}-\sqrt{x}$.

נבדוק את היעקוביאן:

$$\begin{vmatrix} p_x & p_y \\ q_x & q_y \end{vmatrix}=\begin{vmatrix} -\frac{1}{2\sqrt{x}} & \frac{1}{2\sqrt{y}} \\ \frac{1}{2\sqrt{x}} & \frac{1}{2\sqrt{y}} \end{vmatrix}=-\frac{1}{2\sqrt{xy}}\neq 0$$

היעקוביאן שונה מ-0 ולכן נוכל להמשיך:

$$u_x=u_pp_x+u_qq_x=-\frac{1}{2\sqrt{x}}u_p+\frac{1}{2\sqrt{x}}u_q=\frac{1}{2\sqrt{x}}(u_q-u_p)$$

$$u_{xx}=-\frac{1}{4\sqrt{x^3}}(u_q-u_p)+\frac{1}{2\sqrt{x}}(u_{pq}p_x+u_{qq}q_x-u_{pp}p_x-u_{pq}q_x)=-\frac{1}{4\sqrt{x^3}}(u_q-u_p)+\frac{1}{4x}(u_{qq}-2u_{pq}+u_{pp})$$

$$u_y=u_pp_y+u_qq_y=\frac{1}{2\sqrt{y}}u_p+\frac{1}{2\sqrt{y}}u_q=\frac{1}{2\sqrt{y}}(u_p+u_q)$$

$$u_{yy}=-\frac{1}{4\sqrt{y^3}}(u_p+u_q)+\frac{1}{2\sqrt{y}}(u_{pp}p_y+u_{pq}q_y+u_{pq}p_y+u_{qq}q_y)=-\frac{1}{4\sqrt{y^3}}(u_p+u_q)+\frac{1}{4y}(u_{pp}+2u_{pq}+u_{qq})$$

נציב בחזרה במשוואה לקבל:

$$xu_{xx}-yu_{yy}+\frac{1}{2}(u_x-u_y)=0$$

$$-\frac{1}{4\sqrt{x}}(u_q-u_p)+\frac{1}{4}(u_{pp}-2u_{pq}+u_{qq})+\frac{1}{4\sqrt{y}}(u_p+u_q)-\frac{1}{4}(u_{pp}-2u_{pq}+u_{qq})$$

$$+\frac{1}{2}\left(\frac{1}{2\sqrt{x}}(u_q-u_p)-\frac{1}{2\sqrt{y}}(u_p+u_q)\right)=0$$

$$u_{pq}=0$$

מקרה 2: $\Delta = xy < 0$ ולכן אליפטי.
ע"פ המשוואה האופיינית נקבל:

$$\frac{dy}{dx} = \frac{\pm\sqrt{xy}}{x} = \pm\sqrt{\frac{y}{x}}$$

ע"י פתירת המד"ר נקבל $c_{1,2} = \sqrt{y} \pm x\sqrt{\ }$ ולכן נקבל $p(x,y) = \sqrt{y},\ q(x,y) = \sqrt{x}$. נבדוק את היעקוביאן:

$$\begin{vmatrix} p_x & p_y \\ q_x & q_y \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2\sqrt{y}} \\ \frac{1}{2\sqrt{x}} & 0 \end{vmatrix} = -\frac{1}{4\sqrt{xy}} \neq 0$$

היעקוביאן שונה מ-0 ולכן נוכל להמשיך:

$$u_x = u_p p_x + u_q q_x = \frac{1}{2\sqrt{x}} u_q$$

$$u_{xx} = -\frac{1}{4\sqrt{x^3}} u_q + \frac{1}{2\sqrt{x}}\left(u_{pq} p_x + u_{qq} q_x\right) = -\frac{1}{4\sqrt{x^3}} u_q + \frac{1}{4x} u_{qq}$$

$$u_y = u_p p_y + u_q q_y = \frac{1}{2\sqrt{y}} u_p$$

$$u_{yy} = -\frac{1}{4\sqrt{y^3}} u_p + \frac{1}{2\sqrt{y}}\left(u_{pp} p_y + u_{pq} q_y\right) = -\frac{1}{4\sqrt{y^3}} u_p + \frac{1}{4y} u_{pp}$$

נציב בחזרה במשוואה לקבל:

$$x u_{xx} - y u_{yy} + \frac{1}{2}\left(u_x - u_y\right) = 0$$

$$-\frac{1}{4\sqrt{x}} u_q + \frac{1}{4} u_{qq} + \frac{1}{4\sqrt{y}} u_p - \frac{1}{4} u_{pp} + \frac{1}{2}\left(\frac{1}{2\sqrt{x}} u_q - \frac{1}{2\sqrt{y}} u_p\right) = 0$$

$$u_{qq} - u_{pp} = 0$$